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TECHNICAL MEMORANDUM

AN ANALYTIC SOLUTION TO THE SOUND PRESSURE FIELD RESULTING FROM A PLANE WAVE INCIDENT ON A CYLINDER AND CONCENTRIC CYLINDRICAL SECTION

Prepared for

The Bureau of Ships
Code 689B

14 March 1963

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This technical memorandum contains partial results obtained during an analytical study of the sound field near a dome-baffle-transducer complex.

14 March 1963

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I. INTRODUCTION

This memorandum describes a method for computing the solution to the boundary value problem consisting of a plane sound wave incident on an infinite-length right circular cylinder and a section of a right circular cylindrical shell concentric with the cylinder. The problem is formulated so that the section of the cylindrical shell shields the circular cylinder from the incident plane wave. The frequency of the incident wave, the surface impedance of the cylinder and shell, the relative spacing of the cylinder and shell and the angular extent of the shell are parameters of the solution. This boundary value problem has been considered because it represents a tractable formulation for investigating the sound field in the vicinity of a sonar transducer, produced by a source, such as the ship's screws, after this sound field has been modified by a baffle interposed between the source and the transducer. The cylinder represents the transducer and the section of a cylindrical shell represents the baffle.

The material in this memorandum is concerned primarily with the development of a formal solution to the problem as formulated above. Justification for applying this mathematical model to the baffle-transducer complex and a determination of the range of parameters which can be usefully considered will be considered in a subsequent memorandum.

II. FORMULATION OF THE PROBLEM

Consider the problem, shown schematically in Figure 1, of a plane sound wave incident on a right circular cylinder of infinite length and radius a, and a section of a right circular cylindrical shell, also of infinite length, located at radius b, where $b > a$. The shell, which will be called the baffle in the following, has an angular width $2(\pi - \varphi)$ and is positioned with respect to the incident wave so that symmetry in the angle θ is preserved. (The case in which symmetry in the angle θ is not preserved is discussed in Section V.) The thickness of the baffle is assumed to be negligible. It is desired to obtain a solution to the wave equation which describes the total sound field throughout all of space external to the cylinder for the incident plane wave and the scattered and diffracted waves arising from specified boundary conditions on the surface of the cylinder at radius a and on the inner and outer surface of the baffle at radius b.

As is well known, the wave equation in cylindrical coordinates for cases in which the solution is not a function of axial position* is

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{1}$$

where p is the sound pressure, r and θ are the independent coordinates, t is the time and c is the speed of sound. The general solution to this equation, which can be obtained by assuming that the solution has the form of a product of functions of each of the independent coordinates and the time, is, for a

*In this boundary value problem the baffle and cylinder are assumed to be of infinite length in order that the end effects may be neglected. The solution is therefore independent of axial position.

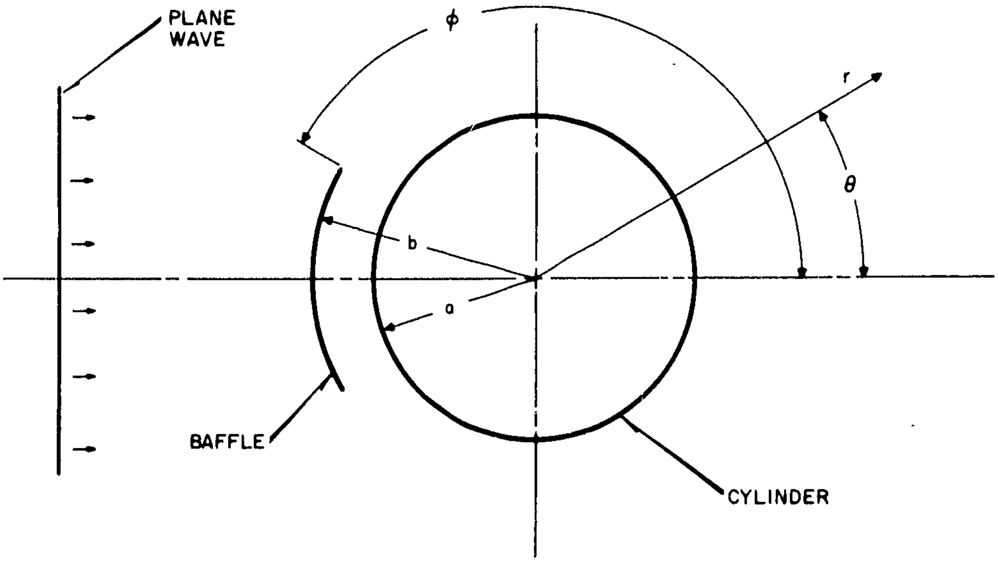

Fig. 1 – GEOMETRY OF THE INFINITE-LENGTH CYLINDER AND CYLINDRICAL SHELL

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sinusoidal time dependence,[1]

$$p = \sum_{n=0}^{\infty} [A_n J_n(kr) + B_n N_n(kr)] \cos n\theta \, e^{-i\omega t} \tag{2}$$

where A_n and B_n are undetermined constants, $J_n(kr)$ and $N_n(kr)$ are the Bessel and Neumann functions respectively, (i.e., the two independent solutions to Bessel's equation for integral orders[2]), ω is the angular frequency of the incident wave, $k = 2\pi/\lambda$ is the wave number and λ is the wavelength of the incident wave. In order to obtain a better physical insight into the meaning of this solution, it can be rewritten in the form[3]

$$p = \sum_{n=0}^{\infty} [C_n H_n^{(1)}(kr) + D_n H_n^{(2)}(kr)] \cos n\theta \, e^{-i\omega t}, \tag{3}$$

where $H_n^{(1)}(kr) = J_n(kr) + i\,N_n(kr)$ is the Hankel function of the first kind and represents a cylindrical wave propagating radially outward; and $H_n^{(2)}(kr) = J_n(kr) - i\,N_n(kr)$ is the Hankel function of the second kind and represents a cylindrical wave propagating radially inward[3], corresponding to reflected and incident cylindrical waves respectively. The radial particle velocity is obtained from the pressure according to the relation

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[1]See e.g., P. M. Morse, Vibration and Sound, McGraw-Hill, New York, 1948, p. 298.

[2]Ibid. pp. 188, 196.

[3]P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York, 1953, p. 1371.

$$i\rho c k \; v = \frac{\partial p}{\partial r} \, , \tag{4}$$

where v is the radial particle velocity, $i = \sqrt{-1}$, ρ is the density of the medium and the remaining terms have been previously defined.

Before proceeding to an evaluation of the undetermined constants so that the specified boundary conditions are satisfied, it is of interest to recall briefly several fundamental identities and to discuss some notation pertinent to representing boundary conditions on the baffle. Consider a function $f(\theta)$ in the form of equations (2) and (3).

$$\begin{aligned} f(\theta) &= \sum_{n=0}^{\infty} A_n(r) \cos n\theta, && 0<|\theta|<\varphi \\ &= 0 && \varphi<|\theta|<\pi \end{aligned} \tag{5}$$

The function $f(\theta)$ can be written as a single expression valid for all θ by expanding it in a Fourier series. The Fourier series expansion of $f(\theta)$ results in the identity

$$f(\theta) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} A_j(r) \begin{Bmatrix} n \\ j \end{Bmatrix} \cos n\theta, \quad 0<|\theta|<\pi \tag{6}$$

where $\epsilon_o = 1$, $\epsilon_n = 2$ when $n \neq 0$, and

$$\begin{Bmatrix} n \\ j \end{Bmatrix} = \frac{\sin |n-j| \varphi}{|n-j| \varphi} + \frac{\sin (n+j) \varphi}{(n+j) \varphi}$$

Here the Fourier coefficients have been determined in the usual manner,[4] i.e.,

$$\frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$$

which in this case reduces to

$$\frac{2}{\pi}\int_0^{\varphi}\left[\sum_{j=0}^{\infty} A_j(r)\cos j\theta\right]\cos n\theta \, d\theta = \frac{\epsilon_n}{2\pi}\,\varphi\sum_{j=0}^{\infty} A_j(r)\begin{Bmatrix} n \\ j \end{Bmatrix} .$$

As indicated in eq. (6) this expansion is valid for all θ.

The expression for $f(\theta)$ given in eq. (6) may also be obtained as the product

$$f(\theta) = g(\theta)h(\theta), \tag{7}$$

where

$$g(\theta) = 1, \qquad 0<|\theta|<\varphi$$

$$= 0, \qquad \varphi<|\theta|<\pi$$

and

$$h(\theta) = \sum_{n=0}^{\infty} A_n(r)\cos n\theta, \qquad 0<|\theta|<\pi .$$

[4] See e.g., D. Jackson, Fourier Series and Orthogonal Polynomials Mathematical Association of America, 1941, p. 7.

The Fourier expansion of $g(\theta)$ is

$$g(\theta) = \sum_{n=0}^{\infty} \frac{\epsilon_n}{2\pi} \varphi \frac{\sin n\varphi}{n\varphi} \cos n\theta \tag{8}$$

The equivalent of eq. (6) for the situation when

$$\begin{aligned} f(\theta) &= 0, && 0<|\theta|<\varphi \\ &= \sum_{n=0}^{\infty} B_n(r) \cos n\theta, && \varphi<|\theta|<\pi \end{aligned} \tag{9}$$

is

$$f(\theta) = \sum_{n=0}^{\infty} \left[B_n(r) - \frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} B_j(r) \begin{Bmatrix} n \\ j \end{Bmatrix} \right] \cos n\theta. \tag{10}$$

Equation (10) can also be obtained as the product

$$f(\theta) = g(\theta)h(\theta), \tag{11}$$

where

$$\begin{aligned} g(\theta) &= 0, && 0<|\theta|<\varphi \\ &= 1, && \varphi<|\theta|\pi \end{aligned}$$

and

$$h(\theta) = \sum_{n=0}^{\infty} B_n(r) \cos n\theta, \qquad 0<|\theta|<\pi.$$

It is convenient to develop the solution for two different geometric regions. Let Region 1 be defined by $a<r<b$, $0<|\theta|<\pi$, and let Region 2 be defined by $r>b$, $0<|\theta|<\pi$. In Region 2 the pressure p_2 can be written as the sum of the incident plane wave plus a general set of cylindrical waves traveling radially outward. Thus

$$p_2(r,\theta) = \sum_{n=0}^{\infty} [\epsilon_n i^n J_n(kr) + A_n H_n^{(1)}(kr)]\cos n\theta \; e^{-i\omega t}, \tag{12}$$

where the first term under the summation is the expression, in cylindrical coordinates, of a plane wave[5] traveling as shown in Figure 1, $\epsilon_o = 1$, $\epsilon_n = 2$ when $n \neq 0$, and $i = \sqrt{-1}$. In Region 1 the pressure p_1 can be written as the sum of a general set of cylindrical waves which are propagating radially inward and a set which are propagating radially outward as given by eq. (3). Thus

$$p_1(r,\theta) = \sum_{n=0}^{\infty} \left[B_n H_n^{(1)}(kr) + C_n H_n^{(2)}(kr)\right] \cos n\theta \; e^{-i\omega t}. \tag{13}$$

At the boundary $r = a$, i.e. on the surface of the cylinder, either $p_1(a,\theta)$ or the radial particle (or medium) velocity $v_1(a,\theta)$ is a prescribed function. On the closed portion of the boundary on the baffle at $r = b$, i.e., for $\varphi<|\theta|<\pi$, either $p_1(b,\theta)$ or $v_1(b,\theta)$ is a prescribed function, while on the open portion of the

[5] P. M. Morse, <u>loc. cit.</u>, p. 347.

boundary at $r = b$, i.e. for $0<|\theta|<\varphi$, both the pressure $p_1(b,\theta)$, and the radial particle velocity $v_1(b,\theta)$, must equal the pressure $p_2(b,\theta)$, and velocity $v_2(b,\theta)$, respectively, in order to establish continuity in the pressure and radial particle velocity between Regions 1 and 2.

For the pressure and velocity in Region 1 the boundary conditions at the radius $r = b$ can be written as

$$p_1(b,\theta) = \left\{ p_1(b,\theta) - [p_1(b,\theta)]_{0<|\theta|<\varphi} \right\} + [p_2(b,\theta)]_{0<|\theta|<\varphi}. \tag{14}$$

$$v_1(b,\theta) = \left\{ v_1(b,\theta) - [v_1(b,\theta)]_{0<|\theta|<\varphi} \right\} + [v_2(b,\theta)]_{0<|\theta|<\varphi} \tag{15}$$

where the brackets, [], denote a Fourier expansion as given by eq. (6), and the quantities without brackets are applicable for all θ. Braces, $\{\}$, are used to indicate the terms which represent a Fourier expansion as given by eq. (10). The corresponding equations at the radius $r = b$ for the pressure and velocity in Region 2 are

$$p_2(b,\theta) = \left\{ p_2(b,\theta) - [p_2(b,\theta)]_{0<|\theta|<\varphi} \right\} + [p_1(b,\theta)]_{0<|\theta|<\varphi} \tag{16}$$

$$v_2(b,\theta) = \left\{ v_2(b,\theta) - [v_2(b,\theta)]_{0<|\theta|<\varphi} \right\} + [v_1(b,\theta)]_{0<|\theta|<\varphi} \tag{17}$$

The constants A_n, B_n and C_n will be evaluated for a specific set of boundary conditions in Section III, corresponding to a physical situation in which the outside of the baffle is perfectly rigid, i.e., $v_2(b,\theta) = 0$ for $\varphi<|\theta|<\pi$; the inside of the baffle is soft, i.e., $p_1(b,\theta) = 0$ for $\varphi<|\theta|<\pi$; and the cylinder is rigid, i.e., $v_1(a,\theta) \equiv 0$. In Section IV the solution will be evaluated for more general, non-zero values at the boundaries.

III. EVALUATION OF THE UNDETERMINED CONSTANTS

The undetermined constants, A_n, B_n, and C_n must be evaluated in order to satisfy the boundary conditions specified on the cylinder and both faces of the baffle. The case to be considered in this section is that of a rigid outer baffle surface, a soft inner baffle surface, and a rigid cylinder surface. A formal statement of these boundary conditions is:

$$v_2(r,\theta)\Big|_{r=b} = v_1(r,\theta)\Big|_{r=b}\,, \qquad 0<|\theta|<\varphi \tag{18}$$

$$= 0\,, \qquad \varphi<|\theta|<\pi$$

$$p_1(r,\theta)\Big|_{r=b} = p_2(r,\theta)\Big|_{r=b}\,, \qquad 0<|\theta|<\varphi \tag{19}$$

$$= 0\,, \qquad \varphi<|\theta|<\pi$$

$$v_1(r,\theta)\Big|_{r=a} = 0\,, \qquad 0<|\theta|<\pi \tag{20}$$

The solutions for Regions 1 and 2 can be separated into Bessel and Neumann functions. Equations (12) and (13)* then become

$$p_2 = \sum_{n=0}^{\infty}\left[\epsilon_n i^n J_n(kr) + A_n J_n(kr) + iA_n N_n(kr)\right]\cos n\theta \tag{21}$$

*The time component $e^{-i\omega t}$ of the solution will henceforth be suppressed.

and

$$p_1 = \sum_{n=0}^{\infty} \left[(C_n + D_n)\, J_n(kr) + i(C_n - D_n)\, N_n(kr)\right] \cos n\theta \quad (22)$$

Without loss of generality, eq. (22) can be rewritten as

$$p_1 = \sum_{n=0}^{\infty} [E_n J_n(kr) + iF_n N_n(kr)] \cos n\theta \quad (23)$$

where $C_n + D_n = E_n$ and $(C_n - D_n) = F_n$.

Consider the boundary condition given in eq. (20). The radial particle velocity for Region 1 is obtain by using eq. (4).

$$i\rho c k v_1\Big|_{r=a} = \sum_{n=0}^{\infty} k[E_n J_n'(ka) + iF_n N_n'(ka)] \cos (n\theta) \equiv 0 \quad (24)$$

where the primes indicate differentiation with respect to the argument kr and the Bessel and Neumann functions have been evaluated at $r = a$. Equation (24) implies that

$$F_n = i\, E_n \frac{J_n'(ka)}{N_n'(ka)} . \quad (25)$$

Thus, the solution in Region 1 can be written as

$$p_1 = \sum_{n=0}^{\infty} E_n \left[J_n(kr) - \frac{J_n'(ka)}{N_n'(ka)} N_n(kr)\right] \cos n\theta . \quad (26)$$

Consider now the substitution of the boundary conditions, equations (18) and (19), into the identities given in equations

(14) and (17). These equations reduce to

$$p_1(b,\theta) = [p_2(b,\theta)]_{0<|\theta|<\varphi} \tag{27}$$

and

$$v_2(b,\theta) = [v_1(b,\theta)]_{0<|\theta|<\varphi} \tag{28}$$

The identities expressed in equations (14) and (17) were chosen in this case because the quantities

$$\left\{p_1(b,\theta) - [p_1(b,\theta)]_{\varphi<|\theta|<\varphi}\right\} \text{ and } \left\{v_2(b,\theta) - [v_2(b,\theta)]_{0<|\theta|<\varphi}\right\}$$

were specified as boundary conditions. Equations (14) and (17) therefore lead to a direct determination of A_n and E_n. If, for example, the velocity were specified on the inside of the baffle and the pressure were specified on the outside of the baffle, then equations (15) and (16) would be more advantageous in solving for the undetermined constants A_n and E_n.

The quantities

$$[p_2(b,\theta)]_{0<|\theta|<\varphi} \text{ and } [v_1(b,\theta)]_{0<|\theta|\varphi}$$

are determined according to eq. (6) using the forms of the solution for Regions 1 and 2 given in equations (21) and (26). Equations (27) and (28) then become

$$\sum_{n=0}^{\infty} E_n\left[J_n(kb) - \frac{J_n'(ka)}{N_n'(ka)} N_n(kb)\right]\cos n\theta = \sum_{n=0}^{\infty} \frac{\epsilon_n}{2\pi}\varphi \sum_{j=0}^{\infty}\Big\{\epsilon_j i^j J_j(kb)$$

$$+ A_j J_j(kb) + iA_j N_j(kb)\Big\}\left\{\begin{matrix} n \\ j \end{matrix}\right\} \cos n\theta \tag{29}$$

and

$$\sum_{n=0}^{\infty} \frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} \left\{ E_j \left[J_j'(kb) - \frac{J_j'(ka)}{N_j'(ka)} N_j'(kb) \right] \right\} \begin{Bmatrix} n \\ j \end{Bmatrix} \cos n\theta =$$

$$\sum_{n=0}^{\infty} \left[i^n \epsilon_n J_n'(kb) + A_n J_n'(kb) + i A_n N_n'(kb) \right] \cos n\theta \tag{30}$$

Equations (29) and (30) are valid for all θ and must therefore be satisfied term by term. Consequently,

$$E_n \left[J_n(kb) - \frac{J_n'(ka)}{N_n'(ka)} N_n(kb) \right] =$$

$$\frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} \left\{ \epsilon_j i^j J_j(kb) + A_j J_j(kb) + i A_j N_j(kb) \right\} \begin{Bmatrix} n \\ j \end{Bmatrix}, \tag{31}$$

and

$$\frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} E_j \left[J_j'(kb) - \frac{J_j'(ka)}{N_j'(ka)} N_j'(kb) \right] \begin{Bmatrix} n \\ j \end{Bmatrix} =$$

$$\epsilon_n i^n J_n'(kb) + A_n J_n'(kb) + i A_n N_n'(kb) \tag{32}$$

Equation (32) is now solved for A_n to yield

$$A_n = \frac{1}{J_n'(kb) + iN_n'(kb)} \left\{ \frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} E_j \left[J_j'(kb) - \frac{J_j'(ka)}{N_j'(ka)} N_j'(kb) \right] \left\{ \begin{matrix} n \\ j \end{matrix} \right\} - \epsilon_n i^n J_n'(kb) \right\} \tag{33}$$

This value of A_n is substituted into eq. (31) to yield

$$E_n \left[J_n(kb) - \frac{J_n'(ka)}{N_n'(ka)} N_n(kb) \right] =$$

$$\frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} \left\{ \epsilon_j i^j J_j(kb) + \frac{J_j(kb) + iN_j(kb)}{J_j'(kb) + iN_j'(kb)} \left[\frac{\epsilon_j}{2\pi} \varphi \sum_{\ell=0}^{\infty} E_\ell \cdot \left\{ J_\ell'(kb) - \frac{J_\ell'(ka)}{N_\ell'(ka)} N_\ell'(kb) \right\} \left\{ \begin{matrix} j \\ \ell \end{matrix} \right\} - \epsilon_j i^j J_j'(kb) \right] \right\} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \tag{34}$$

The order of summation of eq. (34) can be interchanged and the equation rearranged as follows:

$$E_n \left[J_n(kb) - \frac{J_n'(ka)}{N_n'(ka)} N_n(kb) \right] =$$

$$\frac{\epsilon_n}{2\pi} \varphi \sum_{\ell=0}^{\infty} \left[E_\ell \left\{ J_\ell'(kb) - \frac{J_\ell'(ka)}{N_\ell'(ka)} N_\ell'(kb) \right\} \sum_{j=0}^{\infty} \left\{ \begin{matrix} j \\ \ell \end{matrix} \right\} \frac{J_j(kb) + iN_j(kb)}{J_j'(kb) + iN_j'(kb)} \frac{\epsilon_j}{2\pi} \varphi \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \right]$$
$$+ \frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} \left\{ \epsilon_j i^j J_j(kb) - \epsilon_j i^j J_j'(kb) \frac{J_j(kb) + iN_j(kb)}{J_j'(kb) + iN_j'(kb)} \right\} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \tag{35}$$

The summations over the index j involve known constants only (the Bessel and Neumann functions) so that the index j can be summed out. Equation (35) therefore can be written in the form

$$E_n - \sum_{\ell} \alpha_{\ell n} E_{\ell} = \beta_n, \tag{35a}$$

which is an infinite set of inhomogeneous equations relating the E's to each other.

A qualitative examination of the functional forms of the quantities $\alpha_{\ell n}$ and β_n shows that the factors $\left\{ {j \atop \ell} \right\}$ and $\left\{ {n \atop j} \right\}$ will have a tendency to cause the major contribution to the sum on ℓ to arise for values of ℓ in the vicinity of n. One would expect, therefore, that it should be possible to obtain accurate numerical values for the first N of the E's if the indices in the summation (n and ℓ) in eq. (35a) are restricted to a finite number $M > N$. If the indices are restricted in this manner, eq. (35a) becomes a set of M inhomogeneous equations in M unknowns, which can be evaluated numerically. The pressure can then be calculated from eq. (26). The number, N, of accurate E's required is determined by the convergence rate of eq. (26). The number M of E's which must be included in eq. (35) to obtain accurate values for the first N of the E's is determined by the convergence effects of the factors $\left\{ {j \atop \ell} \right\}$ and $\left\{ {n \atop j} \right\}$ in computing the sums over the index ℓ.

The qualitative aspects of obtaining numerical values for the pressure described in the preceding paragraph have been examined quantitatively and have been found to be substantially correct. The complete computational procedure as well as some numerical results will be described in a subsequent memorandum.

IV. GENERAL BOUNDARY CONDITIONS

In the previous section a solution to the baffle-cylinder sound field was developed for the case of zero radial particle velocity on the outside of the baffle and zero pressure on the inside of the baffle. Solutions for more general boundary conditions on the baffle can also be developed using a procedure identical to that of Section III.

Expressions for the quantities p_1, v_1, p_2, and v_2 on the boundary r = b are given in equations (14), (15), (16) and (17). In these equations the quantities in the braces $\{\}$ represent the specified boundary conditions. Any combination of pressure and radial particle velocity can be specified as boundary conditions on the baffle. For example if the pressure on the outside of the baffle is specified as some function $P_2(\theta)$ and the radial particle velocity on the inside of the baffle is specified as some function $V_1(\theta)$ then equations (15) and (16) become

$$v_1(b,\theta) = \sum_{n=0}^{\infty} \left[\frac{2}{\pi} \int_{\varphi}^{\pi} V_1(\theta) \cos n\theta \, d\theta\right] \cos n\theta + [v_2(b,\theta)]_{0<|\theta|\varphi} \tag{36}$$

and

$$p_2(b,\theta) = \sum_{n=0}^{\infty} \left[\frac{2}{\pi} \int_{\varphi}^{\pi} P_2(\theta) \cos n\theta \, d\theta\right] \cos n\theta + [p_1(b,\theta)]_{0<|\theta|<\varphi} \tag{37}$$

where the series represent the Fourier expansions of $V_1(\theta)$ and $P_2(\theta)$ respectively. Here equations (36) and (37) are analogous to equations (27) and (28) of Section III. The procedure for determining the unknown coefficients A_n and E_n is then identical to that of the previous section.

The solution for the problem using any other combination of boundary conditions, e.g. pressure specified on both the inside and outside of the baffle, is obtained by choosing the appropriate pair of identities (equations (14) - (18)) and proceeding as in Section III.

V. PLANE WAVES OBLIQUELY INCIDENT TO THE BAFFLE-CYLINDER

In Section III a solution was obtained for the boundary value problem consisting of a plane wave normally incident on the baffle-cylinder. (See Figure 1.) In this section the solution will be obtained for the more general case of a plane wave incident on the baffle-cylinder at some angle ψ as shown in Figure 2. The development of the solution for the oblique incidence case closely follows the normal incidence case presented in Section III. The expansion for the normally incident wave in polar coordinates is[6]

$$p = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos n\theta$$

where the symbols are defined in previous sections. By a simple coordinate transformation, the off-axis wave shown in Figure 2 can be written as

$$p = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \left[\cos n\psi \cos n\theta + \sin n\psi \sin n\theta\right]$$

where ψ is the angle between the wave front normal and the $\theta = 0$ axis. For the off-axis case, the solution is no longer symmetric in θ; consequently, both sine and cosine terms must be present in the Fourier expansion of the solution.[7] Thus the forms of the solution in Regions 1 and 2 must be written as

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[6] P.M. Morse, loc. cit. p. 347

[7] See e.g. R. V. Churchill, Fourier Series and Boundary Value Problems, McGraw-Hill Company, New York, 1941, Chapter IV.

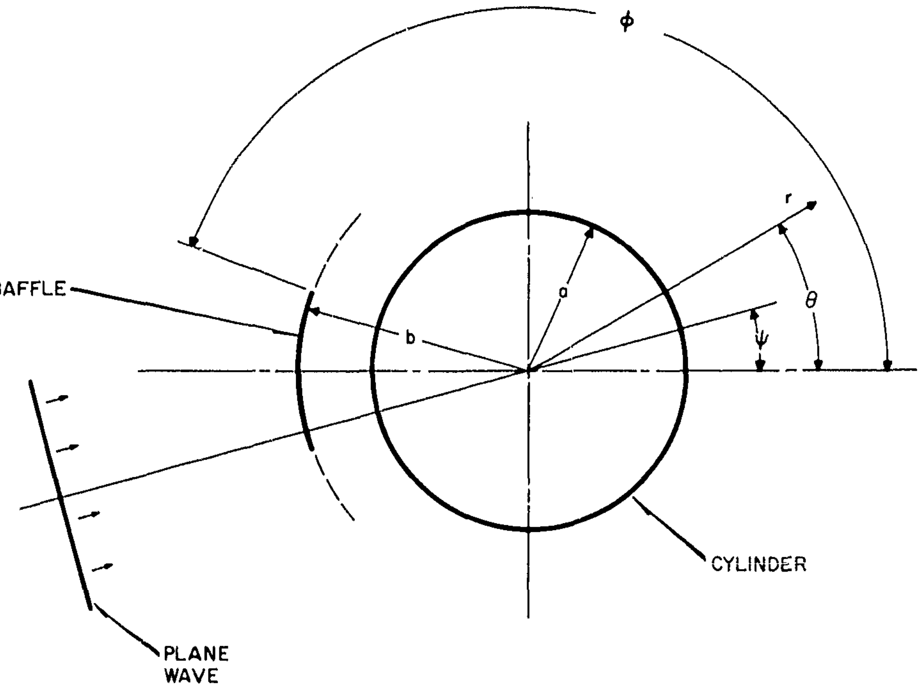

Fig. 2-GEOMETRY OF PLANE WAVE OBLIQUELY INCIDENT ON INFINITE-LENGTH CYLINDER AND CYLINDRICAL SHELL

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$$p_1 = \sum_{n=0}^{\infty} [E_n J_n(kr) + iF_n N_n(kr)]\cos n\theta$$
$$+ \sum_{n=0}^{\infty} [K_n J_n(kr) + iL_n N_n(kr)]\sin n\theta \tag{38}$$

and

$$p_2 = \sum_{n=0}^{\infty} [\epsilon_n i^n J_n(kr)\cos n\psi + A_n J_n(kr) + iA_n N_n(kr)]\cos n\theta$$
$$+ \sum_{n=0}^{\infty} [\epsilon_n i^n J_n(kr)\sin n\psi + G_n J_n(kr) + iG_n N_n(kr)]\sin n\theta . \tag{39}$$

The evaluation of the constants A_n, E_n, F_n, G_n, K_n, and L_n follow the procedure outlined in Section III.

Consider the boundary conditions given by equations (18), (19), and (20). Substitution of equations (38) and (39) into (18), (19), and (20) and performing the operations indicated in Section III yields

$$E_n M_n = \frac{\epsilon_n}{2\pi} \varphi \sum_{\ell=0}^{\infty} \left\{ \epsilon_\ell i^\ell J_\ell(kb)\cos \ell\psi + A_\ell J_\ell(kb) + iA N (kb) \right\} \begin{Bmatrix} n \\ \ell \end{Bmatrix}, \tag{40}$$

$$K_n M_n = \frac{\epsilon_n}{2\pi} \varphi \sum_{\ell=0}^{\infty} \left\{ \epsilon_\ell i^\ell J_\ell(kb)\sin \ell\psi + G_\ell J_\ell(kb) + iG_\ell N_\ell(kb) \right\} \begin{Bmatrix} n \\ \ell \end{Bmatrix}^*, \tag{41}$$

$$\epsilon_n i^n J_n'(kb)\cos n\psi + A_n J_n'(kb) + iA_n N_n'(kb) = \frac{\epsilon_n}{2\pi}\varphi \sum_{\ell=0}^{\infty} \left\{E_\ell M_\ell'\right\}\left\{{n \atop \ell}\right\} , \tag{42}$$

and

$$\epsilon_n i^n J_n'(kb)\sin n\psi + G_n J_n'(kb) + iG_n N_n'(kb) = \frac{\epsilon_n}{2\pi}\varphi \sum_{\ell=0}^{\infty} \left\{K_\ell M_\ell'\right\}\left\{{n \atop \ell}\right\} ,^* \tag{43}$$

where

$$\left\{{n \atop \ell}\right\}^* = \frac{\sin|\ell-n|\varphi}{|\ell-n|\varphi} - \frac{\sin(\ell+n)\varphi}{(\ell+n)\varphi} ,$$

$$M_n = J_n(kb) - \frac{J_{n-1}(ka) - J_{n+1}(ka)}{N_{n-1}(ka) - N_{n+1}(ka)} N_n(kb) ,$$

$$M_n' = J_n'(kb) - \frac{J_{n-1}(ka) - J_{n+1}(ka)}{N_{n-1}(ka) - N_{n+1}(ka)} N_n'(kb) ,$$

and all other symbols are defined as in previous sections. Equations (40) and (42), (41) and (43) can be combined to yield equations similar to eq. (35), i.e.

$$E_n M_n = \frac{\epsilon_n}{2\pi}\varphi \sum_{\ell=0}^{\infty} \left\{{n \atop \ell}\right\}\left\{\epsilon_\ell i^\ell \cos \ell\psi\left[J_\ell(kb) - R_\ell J_\ell'(kb)\right]\right\}$$

$$+ \frac{\epsilon_n}{2\pi}\varphi \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\epsilon_j}{2\pi}\varphi \left\{{n \atop \ell}\right\}\left\{{\ell \atop j}\right\} R_j M_j' E_j , \tag{44}$$

$$K_n M_n = \frac{\epsilon_n}{2\pi} \varphi \sum_{\ell=0}^{\infty} \left\{ {n \atop \ell} \right\}^* \left\{ \epsilon_\ell i^\ell \sin \ell\psi [J_\ell(kb) - R_\ell J_\ell'(kb)] \right\}$$

$$+ \frac{\epsilon_n}{2\pi} \varphi \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \left\{ {n \atop \ell} \right\}^* \left\{ {\ell \atop j} \right\}^* \frac{\epsilon_j}{2\pi} \varphi \, R_j M_j' K_j \quad , \tag{45}$$

where

$$R_j = \frac{J_j(kb) + iN_j(kb)}{J_j'(kb) + iN_j'(kb)} \quad .$$

Equations (44) and (45) can be solved in the same manner as eq. (35) to yield values of E_n and K_n to complete the solution for $p_1(r,\theta)$. For boundary conditions other than those given in equations (18), (19), and (20) the procedure for the treatment of more general boundary conditions given in Section IV is employed.

The solution presented in this section will reduce to the normal incidence case presented in Section III if the angle $\psi = 0$ is substituted into equations (44) and (45). Equation (44) then reduces to eq. (35) of Section III. Equation (45) reduces to an infinite set of homogeneous equations in the unknown K_n which is satisfied only if $K_n=0$, resulting in a solution identical to that of Section III.

Complex incident sound fields can be considered using the solution presented in this section. Assume several plane waves are incident on the baffle-cylinder at angles of incidence ψ_i. The scalar wave equation (eq. 1) is a linear equation; hence, any linear combination of solutions to the wave equation is itself a solution. Therefore the solution to the complex sound field containing M different incident plane waves can be written as

$$p_1 = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \left\{ \left[E_{mn} J_n(kr) + iF_{mn} N_n(kr) \right] \cos n\theta \right. \tag{46}$$

$$\left. + \left[K_{mn} J_n(kr) + iL_{mn} N_N(kr) \right] \sin n\theta \right\} ,$$

$$p_2 = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \left\{ \left[\alpha_m \epsilon_n i^n J_n(kr) \cos n\psi_m + A_{mn} J_n(kr) + iA_{mn} N_n(kr) \right] \cos n\theta \right. \tag{47}$$

$$\left. + \left[\alpha_m \epsilon_n i^n J_n(kr) \sin n\psi_m + G_{mn} J_n(kr) + iG_{mn} N_n(kr) \right] \sin n\theta \right\}$$

where α_m is an amplitude weighting function for the various plane waves. Here it has been assumed that the various incident waves have the same angular frequency. However the solution given in equations (46) and (47) could easily be generalized to include incident plane waves of different frequencies.